SIMULATION OF STRESS AND DEFORMATION BEAM SUBJECTED TO BENDING AND COMPRESSION, FIXED AT ONE END AND AT THE OTHER END SUPPORTED TO HELICAL SPRING

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Abstract - Study stability of elastic beams subjected to compressive force and bending represents an important chapter in the calculation verification and dimensioning of column elements. This paper aims is simulation of stress and deformation beam subjected to bending and compression using the theory of second order (taking into account the effect of deformation of compression force). Also paper aims at simulating by plotting the variation of bending efforts, displacements and section rotation in the particular case of a beam subjected to bending by the transverse force Q acting at a distance \( \alpha L \) (\( \alpha \) variable parameter) and also to compression by the axial force P using professional design software MATHCAD14.

Keywords: simulation buckling bending and compression stress.

1. Introduction

The stability study of elastic beams subjected to bending and compressive forces are solving using the theory of second order (taking into account the effect of deformation of compressive force). The particular case is a straight beam length L with profiled section, fixed at one end and supported elastic at the other. The transverse force Q acting at a distance \( \alpha L \) (\( \alpha \) variable parameter) and the compressive axial force P acting at both ends. The simulation was accomplish using professional design software MATHCAD14 by variation of variable parameter \( \alpha \) (\( \alpha L \) is distance to end up section application of transversal Q) and by variation of variable parameter \( \beta \) (the axial force \( P=\beta P_{cr} \), \( P_{cr} \) buckling critical force).

2. The second order calculus

Consider a same straight bar OA of length L, with profiled section having section area A and axial main moments of inertia Iy (min) and Iz (max). The bar is subjected to bending and axial compression under the influence of a couple of forces N, a transverse force F and uniformly distributed force q. also under the axial forces P (Figure 1).

Figure 1
We note:
- \( w_0 \) and \( \varphi_0 \) are the displacement and rotating of the left end of the bar;
- \( P \), \( T_0 \) and \( M_0 \) - corresponding forces / torques of the left end of the bar.

The second order calculation of deformation beam allows writing equilibrium equations for the deformed shape of the bar subjected to bending and compression force \( P \). The deformations caused by compressive force \( P \) and the bending moment increase with increasing

\[
\frac{d^4}{dx^4} + \frac{P}{E \cdot I_y} \cdot \frac{d^2}{dx^2} = q
\]

The solution differential equation (2) is written:

\[
w(x) = C_1 \sin(kx) + C_2 \cos(kx) + C_3x + C_4 + \bar{w} \\
\varphi(x) = kC_1 \cos(kx) - kC_2 \sin(kx) + C_4 + \frac{d\bar{w}}{dx}
\]

If we derives twice the differential equation (3) we obtained the bending moment and shear functions:

\[
M_i(x) = -EI \left(-k^2C_1 \sin(kx) - k^2C_2 \cos(kx) + \frac{d^2\bar{w}}{dx^2}\right)
\]

\[
T(x) = EI \left(k^2C_1 \cos(kx) - k^2C_2 \sin(kx) - \frac{d^2\bar{w}}{dx^2}\right)
\]

where: \( k \) is a parameter of the compressive force \( P \) and bending stiffness \( E \cdot I_y \);

\[
k^2 = \frac{P}{E \cdot I_y}
\]

\( C_1, C_2, C_3, C_4 \) are the constants determined from the boundary conditions of deformed bar (displacements and / or rotations of the bar ends);

\[\bar{w}(x), \bar{\varphi}(x) = \frac{d\bar{w}}{dx}\] are the particular solutions suitable for cross external loads \( N, F, q \) is written [3]:

\[
\bar{w}_N(x) = -\frac{N}{P} \left(1 - \cos(k \cdot (x-a))\right) \ 	ext{for} \ x \geq a
\]

\[
\bar{w}_F(x) = \frac{F}{kP} \left(k \cdot (x-b) - \sin(k \cdot (x-b))\right) \ 	ext{for} \ x \geq b
\]

\[
\bar{w}_q(x) = \frac{q}{kP} \int_{d}^{x} \left(k \cdot (x-t) - \sin(k \cdot (x-t))\right)dt \ 	ext{for} \ x \in (c, d)
\]

If the origin parameter values: \( w_0, \varphi_0, M_0 \) and \( T_0 \) are known, the boundary conditions are written take into account the relationship (3) and (4) [3]:

\[
w(0) = w_0 = C_2 + C_4 \\
\varphi(0) = \varphi_0 = kC_1 + C_3 \\
M_i(0) = M_0 = E \cdot I_y (k^2 \cdot C_2) \\
T(0) = T_0 + P \cdot \varphi_0 = E \cdot I_y (k^3 \cdot C_1)
\]

The expressions constants \( C_1, C_2, C_3, C_4 \) are obtained [3]:

\[
d\frac{d^2}{dx^2} = -q + \frac{P}{E \cdot I_y} \cdot \frac{d^2}{dx^2}
\]
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\[ C_1 = \frac{T_0 + \phi_0}{kP}, \quad C_2 = \frac{M_0}{EI_y k^2}, \quad C_3 = \frac{T_0}{P}, \quad C_4 = w_0 - \frac{M_0}{P} \]

Substituting expressions functions (8) into (3) and (4) we obtained the solutions:

\[ w(x) = w_0 + \frac{\phi_0}{k} \sin(k \cdot x) - \frac{M_0}{P} \left( \frac{L}{k} \cos(k \cdot x) \right) - \frac{T_0}{k \cdot P} \left( k \cdot x - \sin(k \cdot x) \right) + \bar{w} \]

\[ \phi(x) = \phi_0 \cdot \cos(k \cdot x) - \frac{M_0}{P} k \cdot \sin(k \cdot x) - \frac{T_0}{P} \left( 1 - \cos(k \cdot x) \right) + \frac{d\bar{w}}{dx} \]

\[ M_i(x) = EI_y \left( \phi_0 \cdot k \cdot \sin(k \cdot x) + \frac{M_0}{P} k^2 \cdot \cos(k \cdot x) + \frac{T_0}{P} k \cdot \sin(k \cdot x) - \frac{d^2 \bar{w}}{dx^2} \right) \]

3. Particular case study

We consider a straight bar of length \( L = 5 \) m with profiled section 118 (STAS 565-80) having cross section area \( A = 27.9 \) cm\(^2\) and axial moments of inertia \( I_y = 81.3 \) cm\(^4\), \( I_z = 145 \) cm\(^4\). The bar is fixed at one end and supported elastic at the other, is subject to bending under the action of transverse \( Q = 1 \) kN applied to a L embedding distance and compression by the axial force \( P = \beta P_c \) (buckling critical force) according to different parameter values \( \omega = 0.1 \ldots 0.9 \) (Figure 2). Cylindrical helical spring has the following characteristics: \( d = 18 \) mm, \( D = 100 \) mm, elastic shear modulus \( G = 85 \times 10^3 \) MPa , \( n = 9 \) turns; the straight bar is made of a material having mechanical characteristics: elastic longitudinal modulus \( E = 2.1 \times 10^5 \) MPa; coefficient thinness \( \lambda_0 = 105 \); yield stress \( \sigma_y = 150 \) MPa. The required task:

a) to represent the variation of bending moments, displacements and rotations after second order theory according to different parameter values \( \omega = 0.1 \ldots 0.9 \);

b) to find the distance to end \( \alpha L \) up to section of acting force \( Q \) which displacement is the maximum

c) to represent the variation of moments, displacements and rotations after second order theory according to different parameter values \( \beta = 0.1 \ldots 0.5 \);

![Figure 2](image.png)

The reaction forces \( M_0, T_0 \) and \( V_0 \) after second order theory are obtained from equilibrium equations of mechanics, and boundary condition of displacement for this case:

\[ w(L) = V_A \cdot \delta_A, \quad M_y(L) = 0 \]  

The function (9) of displacements \( w(x) \), rotation \( \phi(x) \), and bending moment \( M_i(x) \), writing with step function \( \Phi \) of MATHCAD software for this case are:

\[ w(x) = -\frac{T_0}{k \cdot P} \left( k \cdot x - \sin(k \cdot x) \right) - \frac{M_0}{P} \left( \frac{L}{k} \cos(k \cdot x) \right) + \frac{Q}{k \cdot P} \left( k \cdot (x - \alpha L) \right) \cdot \Phi(x - \alpha L) \]

\[ \phi(x) = -\frac{T_0}{P} \left( 1 - \cos(k \cdot x) \right) - \frac{M_0}{P} k \cdot \sin(k \cdot x) + \frac{Q}{P} \left( 1 - \cos(k \cdot (x - \alpha L)) \right) \cdot \Phi(x - \alpha L) \]

\[ M_i(x) = \frac{T_0}{k} \sin(k \cdot x) + M_0 \cdot \cos(k \cdot x) + \frac{V_0}{k} \sin(k \cdot (x - \alpha L)) \cdot \Phi(x - \alpha L) - \frac{Q}{k} \cos(k \cdot (x - \alpha L)) \cdot \Phi(x - \alpha L) \]
The condition (10) lead to expression of reaction $T_0$, $M_0$ and $V_A$

\[
T_0 = -Q \frac{(1 - \cos(\alpha \cdot kL)) \cdot \sin(kL \cdot (1 - \alpha)) - \cos(kL) \cdot (1 - \alpha \cdot kL) \cdot \sin((1 - \alpha) \cdot kL) - \left(\frac{kL}{L}\right)^2 E \cdot I_y \cdot \delta_\alpha}{(1 - \cos(\alpha \cdot kL)) \cdot \sin(kL) - \cos(kL) \cdot \left(\frac{kL}{L}\right)^2 E \cdot I_y \cdot \delta_\alpha}
\]

\[
M_0 = -Q \cdot L \frac{\sin(kL) \cdot (1 - \alpha \cdot kL) \cdot \sin((1 - \alpha) \cdot kL) \cdot \left(\frac{kL}{L}\right)^2 E \cdot I_y \cdot \delta_\alpha}{(1 - \cos(\alpha \cdot kL)) \cdot \sin(kL) - \cos(kL) \cdot \left(\frac{kL}{L}\right)^2 E \cdot I_y \cdot \delta_\alpha}
\]

\[
V_A = Q - T_0
\]

3.1. Simulation of the variation of bending moments, displacements and rotations

In Figures 3…11 are presented the variation of bending moment, displacement and rotation for different increasing parameter values $\alpha=0,1 \ldots 0.9$.

![Fig. 3. Variation of bending moment, displacements and rotations for $\alpha=0.1$](image1)

$(\alpha=0.1)$ $x_{2\text{max}}=2435.9 \text{ mm}$; $v_{2\text{max}}=0.8165 \text{ mm}$

![Fig. 4. Variation of bending moment, displacements and rotations for $\alpha=0.2$](image2)

$x_{2\text{max}}=2524.9 \text{ mm}$; $v_{2\text{max}}=2.8578 \text{ mm}$
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Fig. 5. Variation of bending moment, displacements and rotations for $\alpha=0.3$
$x_{v_{2\max}}=2625.1 \text{ mm}; \ v_{2\max}=5.4856 \text{ mm}$

Fig. 6. Variation of bending moment, displacements and rotations for $\alpha=0.4$
$x_{v_{2\max}}=2740.6 \text{ mm}; \ v_{2\max}=8.0657 \text{ mm}$

Fig. 7. Variation of bending moment, displacements and rotations for $\alpha=0.5$
$x_{v_{2\max}}=2877.6 \text{ mm}; \ v_{2\max}=10.0195 \text{ mm}$
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Fig. 8. Variation of bending moment, displacements and rotations for $\alpha=0.6$
$v_{2\text{max}}=3045.8 \text{ mm}; ~v_{2\text{max}}=10,8839 \text{ mm}$

Fig. 9. Variation of bending moment, displacements and rotations for $\alpha=0.7$
$v_{2\text{max}}=3234.3 \text{ mm}; ~v_{2\text{max}}=10,4003 \text{ mm}$

Fig. 10. Variation of bending moment, displacements and rotations for $\alpha=0.8$
$v_{2\text{max}}=3438.2 \text{ mm}; ~v_{2\text{max}}=8,6689 \text{ mm}$
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Fig. 11. Variation of bending moment, displacements and rotations for \( \alpha = 0.9 \)

\[ v_{1\text{max}} = 3845.3 \text{ mm}; v_{2\text{max}} = 6148.5 \text{ mm} \]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( v_{1\text{max}} )</th>
<th>( v_{2\text{max}} )</th>
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<tr>
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<td>0.6312</td>
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<td>2.1692</td>
<td>2.8578</td>
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<tr>
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<td>4.1102</td>
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<td>5.9993</td>
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<tr>
<td>0.5</td>
<td>7.446</td>
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<td>0.6</td>
<td>8.149</td>
<td>10.8839</td>
</tr>
<tr>
<td>0.7</td>
<td>7.9418</td>
<td>10.4003</td>
</tr>
<tr>
<td>0.8</td>
<td>6.8942</td>
<td>8.6689</td>
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<td>0.9</td>
<td>5.3605</td>
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Fig. 12

<table>
<thead>
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<td>0.58</td>
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<td>8.1183</td>
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<tr>
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<td>8.149</td>
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<tr>
<td>0.61</td>
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<td>0.68</td>
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<td>0.69</td>
<td>8.0035</td>
<td>10.5093</td>
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<tr>
<td>0.7</td>
<td>7.9418</td>
<td>10.4003</td>
</tr>
</tbody>
</table>

Fig. 13
3.2. Simulation of the variation of bending moments, displacements and rotations with $\alpha$ parameter

In Table 3 are presented the variation of displacements for different increasing parameter values $\beta=0.1 \ldots 0.9$;

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$V_{2 \text{max}}$</th>
<th>$x_{2 \text{max}}$</th>
</tr>
</thead>
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<td>0.1</td>
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<td>3131.6</td>
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<td>9.7649</td>
<td>3118.3</td>
</tr>
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<td>0.3</td>
<td>10.8834</td>
<td>3103.7</td>
</tr>
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<td>0.4</td>
<td>12.3665</td>
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</tr>
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<td>0.5</td>
<td>14.4329</td>
<td>3070.1</td>
</tr>
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<td>17.5198</td>
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</tr>
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<td>32.879</td>
<td>3005.5</td>
</tr>
<tr>
<td>0.9</td>
<td>63.5234</td>
<td>2979.2</td>
</tr>
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4. Conclusions

From the numerical results obtained for the particular case presented in this paper it can be concluded that the method of second order is an accurate method and the simulation by increasing of $\alpha$ parameter indicate increasing and decreasing of deformations as shown in Figure 3...11; this result can allow to find the distance to end $\alpha$ up to section of acting force $Q$ which displacement is the maximum (Table 1 – Figure 13 and table 2 – Figure 12);

Simulation by increasing of $\beta$ parameter (Table 3 and Figure 14) indicate how it grow the maximum displacement $V_{2 \text{max}}$ with increasing compression force $P=\beta P_{cr}$, and when the force of compression tends to critical buckling force $P_{cr}$, the maximum displacement $v_{2 \text{max}}$ tends to infinity;

The principle of second order method of calculating compressive - bending for this particular case can be extended to other particular cases such as columns and abutting recessed, articulated or embedded frames, etc. in compression - bending calculation principle is the same.

5. References

[9] Marin, C., Marin A. - Determination of the maximal buckling force for a straight beam with different supporting conditions and symmetrically ends (I) - Romanian Review Precision Mechanics, Optics & Mechatronics, no. 41/2012;